An equivalence relation between morphological dynamics and persistent homology in n-D

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GDMM 2021

Outline

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Dynamics vs. persistence

Domains:

- Dynamics $\in$ Mathematical Morphology
- Topological persistence $\in$ Persistent Homology.

Practical uses:

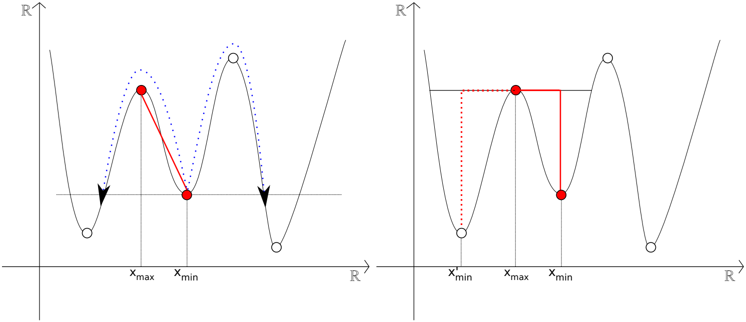
- dynamics $\rightsquigarrow$ markers $\rightsquigarrow$ watersheds $\rightsquigarrow$ segmentation $\rightsquigarrow$ image analysis,
- persistence $\rightsquigarrow$ pairings $\rightsquigarrow$ cancelations $\rightsquigarrow$ simplification $\rightsquigarrow$ image visualisation

Differences:

- dynamics $\rightsquigarrow$ correspond to regional minima,
- persistence $\rightsquigarrow$ is related to gradient vector fields (Morse-Smale complexes).

Both encode topological information of functions.

An example



$\rightsquigarrow$ same pairings but different definitions!

Outline

Morse functions

Morse functions (general definition):

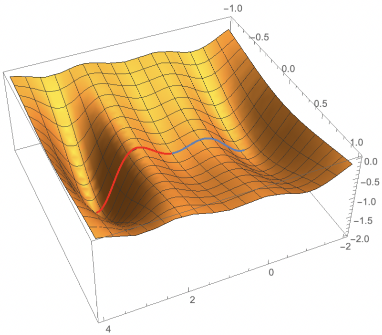
- $f \in \mathcal{C}^2(\mathcal{D})$ and the Hessian matrix is not degenerated at the critical points,
- is does not have any plateau,
- for pairings, we need critical values to be unique.

Dynamics I

- $f : \mathbb{R}^n \to \mathbb{R}$ a Morse function
- $x_{\min}$ a local minimum of f,
- γ a path following the graph of f from $\gamma(0) := x_{\min}$ to $\gamma(1)$ s.t.

$$f(\gamma(1)) < f(x_{\min}),$$

- $\mathrm{effort}(\gamma, x_{\min}) = \max_{s \in [0,1]} f(\gamma(s)) - f(x_{\min}),$
- $\mathrm{dyn}(x_{\min}) := \min_{\forall\ \gamma} \mathrm{effort}(\gamma, x_{\min}).$

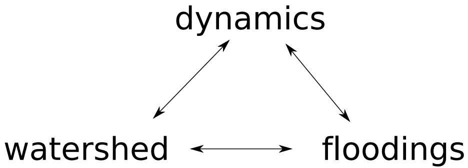


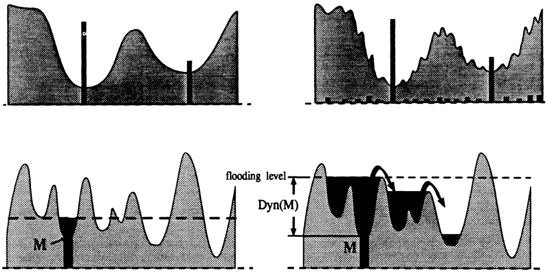
Equivalently, $\mathrm{dyn}(x_{\min}) = f(x_{1\mathrm{sad}}) - f(x_{\min})$ with $x_{1\mathrm{sad}}$ the local max of f corresponding to the minimal effort.

$x_{\min}$ and $x_{1\mathrm{sad}}$ where the optimal path is at maximum height are then paired by dynamics.

Dynamics II

An interesting relation in MM:





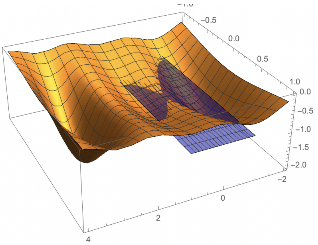
Topological persistence I

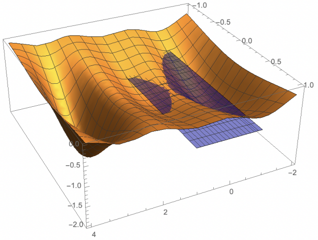
Pairing by persistence

- $f : \mathbb{R}^n \to \mathbb{R}$ a Morse function,
- x_{1sad} a 1-saddle of f,
- $C^{\text{1sad}} = \mathcal{CC}([f \leq f(x_{\text{1sad}})], x_{\text{1sad}})$,
- $\mathcal{CC}_1$ and $\mathcal{CC}_2$ the **two** components of $[f < f(x_{\text{1sad}})]$ whose boundary contains x_{1sad},
- $\text{rep}_i := \arg\min_{x \in \mathcal{CC}_i} f(x)$,
- $x_{\min} := \arg\max_{x \in \{rep_1, rep_2\}} f(x)$,

Then, x_{1sad} is paired with $x_{\min}$ by persistence.

$\Rightarrow$ Topological persistence $:= f(x_{\text{1sad}}) - f(x_{\min})$.

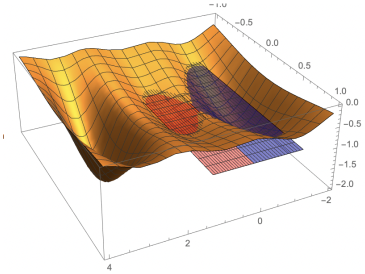




Outline

Hypothesis

- $f : \mathbb{R}^n \to \mathbb{R}$ *a Morse function,*
- $x_{\min}$ *a local minimum of* f,
- $x_{\min}$ *paired with* $x_{1\text{sad}}$ *by dynamics,*
- *We define* $C_1 = CC([f < f(x_{1\text{sad}})], x_{\min})$.
- *We define* C_2 *the component of* $[f < f(x_{1\text{sad}})]$ *which does *not* contain* $x_{\min}$ *and whose closure contains* $x_{1\text{sad}}$.



Property ($P1$)

$x_{\min} = \arg\min_{x \in C_1} f(x)$

Property ($P2$)

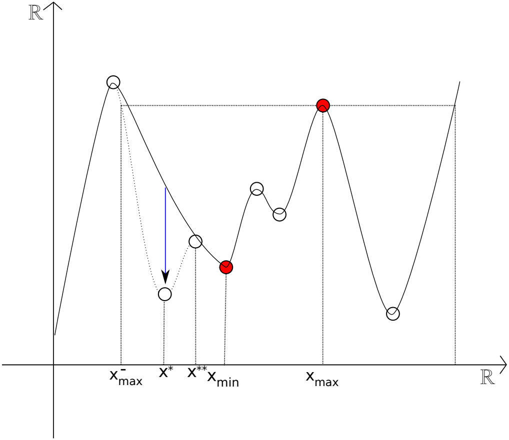
$x'_{\min} := \arg\min_{x \in C_2} f(x)$ *satisfies* $f(x'_{\min}) < f(x_{\min})$.

Theorem

$x_{1\text{sad}}$ *is paired with* $x_{\min}$ *by persistence.*

Property $(P1)$

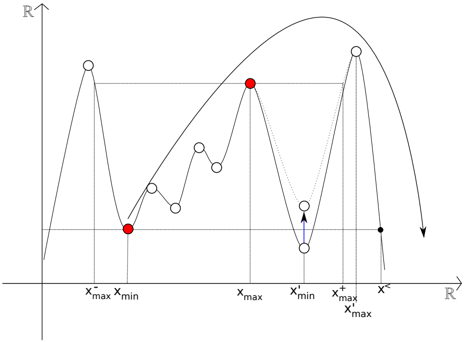
$x_{\min} = \arg\min_{x \in C_1} f(x)$



Intuition: if there exists $x^* \in C_1$ s.t. $f(x^*)$ is lower than $f(x_{\min})$,
$\mathrm{dyn}(x_{\min}) < f(x_{1\mathrm{sad}}) - f(x_{\min}) \rightsquigarrow$ contradiction.

Property ($P2$)

$x'_{min} := \arg\min_{x \in C_2} f(x)$ *satisfies* $f(x'_{min}) < f(x_{min})$.



Intuition: if we increase $f(x'_{min})$ *above* $f(x_{min})$, $\mathrm{dyn}(x_{min})$ increases $\rightsquigarrow$ contradiction.

First main result of this paper:

Theorem

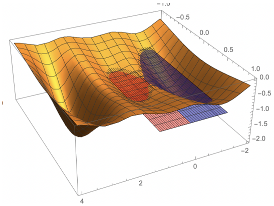
x_{1sad} is paired with $x_{\min}$ by persistence.

Outline

Pairing by persistence implies pairing by dynamics I

Hypothesis

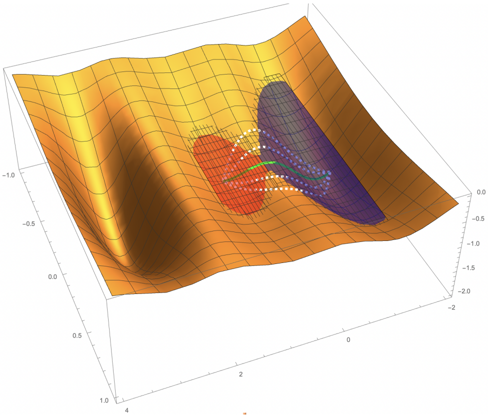
- $f : \mathbb{R}^n \to \mathbb{R}$ *a Morse function with* $x_{1\text{sad}}$ *a* 1*-saddle of* f*,*
- $x_{1\text{sad}}$ *and* $x_{\min}$ *are paired by persistence:*
  - C_1, C_2 *the two components of* $[f < f(x_{1\text{sad}})]$ *whose closure contains* $x_{1\text{sad}}$*,*
  - C_1 *contains* $x_{\min}$ *and* $x_{\min} := \arg\min_{x \in C_1} f(x)$
  - C_2 *does not contain* $x_{\min}$ *and* $x'_{\min} := \arg\min_{x \in C_2} f(x)$,
  - $f(x'_{\min}) < f(x_{\min})$.



Pairing by persistence implies pairing by dynamics II

Property

1. $\exists$ *a descending path* γ *from* $x_{\min}$ *to* $x'_{\min}$ *corresponding to an effort of* $f(x_{1\mathrm{sad}}) - f(x_{\min})$*,*
2. *then* $\mathrm{dyn}(x_{\min}) \leq f(x_{1\mathrm{sad}}) - f(x_{\min})$*,*
3. *to reach a level lower than* $f(x_{\min})$ *on* f *in an optimal way, the only possibility is to go through a* 1*-saddle,*
4. *then any optimal descending path goes through* $x_{1\mathrm{sad}}$*,*
5. *then* $\mathrm{dyn}(x_{\min}) \geq f(x_{1\mathrm{sad}}) - f(x_{\min})$*,*
6. *then* $\mathrm{dyn}(x_{\min}) = f(x_{1\mathrm{sad}}) - f(x_{\min})$*,*
7. *the only local extremum satisfying* (6) *is* $x_{1\mathrm{sad}}$*.*



Pairing by persistence implies pairing by dynamics III

Second main result of this paper:

Theorem

When f is a n-D Morse function and $x_{\min}$ and x_{1sad} are paired by persistence, then x_{1sad} and $x_{\min}$ are paired by dynamics too.

Outline

- Not only two directions are possible as in 1D: this number becomes infinite in 2D and beyond,
- We had to prove that at a 1-saddle point (on a Morse function), we have always two components which merge when the threshold sets increase to $f(x_{\text{1sad}})$,
- We had to change systematically the coordinates so that the functions can be written:

$$f(x_1, \dots, x_n) = -x_1^2 + x_2^2 + \dots + x_n^2.$$

- We had to make “algorithmic” the computation of optimal paths in n-D to prove that they always go through a 1-saddle point,
- the calculus relative to the proofs are a little more complex but the concept is the same.

Outline

Summary:

- on Morse functions, pairings by persistence and by dynamics are equivalent,
- persistence and dynamics values are then equal,
- another relation between MM and MT:

$$WS(f) \cup WS(-f) = MS(f),$$

- finally, we reinforced the relation between MM and MT!

Future works:

- extension to discrete Morse functions (Forman),
- investigate if algorithms of MM can be used in PH and conversely,

Questions

Is this a Morse function?