Hölder Exponents on Metric Spaces using Morphological Operators

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Plan

Fractal dimension and Hölder functions

- Fractals can be used as random models for sets and functions: characterization or simulation
- Fractal sets and fractal functions can be considered under two different viewpoints
    - Global property: Self-similarity (i.e., statistical phenomenon is equal to itself at all the scales)
    - Local property: Non-integer dimension (fractal dimension)

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- Fractal sets and fractal functions can be considered under two different viewpoints
  - Global property: Self-similarity (i.e., statistical phenomenon is equal to itself at all the scales)
  - Local property: Non-integer dimension (fractal dimension)
- Hölder continuous functions: Class of functions which are relevant in fractal analysis (fractal dimension and Hölder exponent of functions are related in many cases)
- Estimates of the dimension and the exponent of this kind of functions are classicaly based either on
  - Wavelet theory
  - Multiscale morphological operators

Fractals and mathematical morphology

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- First contributions dealing with that practical method to connect a morphological measure of local oscillation and fractal dimension focussed on the 1D case (Tricot, 1988,1993) and were then extended to the 2D case (Rigaut, 1988; Maragos, 1993,1994; Soille, 1996)

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- Our goal: Hölder function characterization is revisited from the mathematical morphology viewpoint
    - review the notion of fractal dimension estimation from the mathematical morphology viewpoint
    - review properties of morphological operators on metric spaces for general equicontinuous functions
    - generalization to the metric space case, where morphological multiscale operators on lenght spaces are used to provide Hölder exponent estimation

Hölder continuous functions

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is an α-Hölder continuous function, or Hölderian function, for which it exists the exponent α, $0 < \alpha \leq 1$, and a constant K, when the following condition is satisfied

$$|f(x) - f(y)| \leq K\|x - y\|^{\alpha}, \quad \forall x, y \in \mathbb{R}^n, \quad K > 0$$

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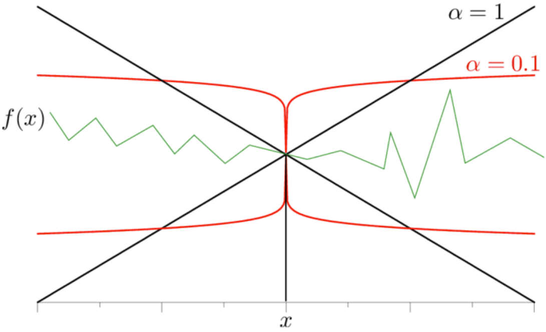
- Let (X, d) be a metric space. The real-valued function $f : X \to \mathbb{R}$ is α-Hölder in X with $0 < \alpha \leq 1$ if there exists $K > 0$ such that

$$|f(x) - f(y)| \leq Kd(x, y)^{\alpha}, \quad \forall x, y \in X$$

Hölder continuous functions

Lipschitz condition vs. Hölder condition



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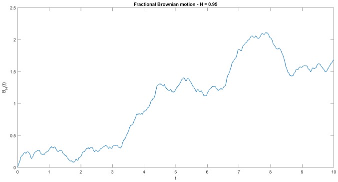
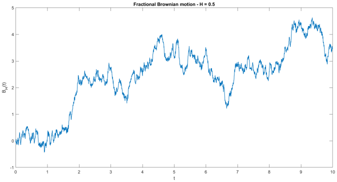
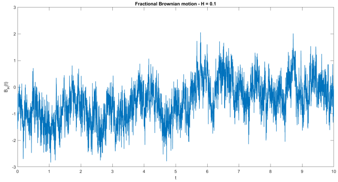


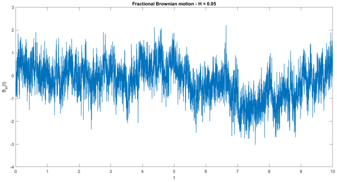
$\alpha = 0.9$



$\alpha = 0.5$



$\alpha = 0.1$



$\alpha = 0.05$

Wavelet transform and Hölder exponent estimation (Mallat, 2008)

- A wavelet is a function ψ with a zero average, i.e., $\int_{-\infty}^{\infty} \psi(x)dx = 0$, and n vanishing moments (multiscale differential operator of order n), i.e., $\int_{-\infty}^{\infty} x^m \psi(x)dx = 0,\ 1 \leq m \leq n$
- The wavelet transform of signal f is defined as

$$Wf(x,s) = f \star \psi_s(x) = \langle f, \psi_{x,s} \rangle = \int_{-\infty}^{\infty} f(z) \frac{1}{\sqrt{s}} \psi \left(\frac{z-x}{s} \right) dz$$

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- The decay of the wavelet transform amplitude across scales is related to the pointwise reglarity of the signal: Measuring the decay is equivalent to zooming into signal structures with a scale that goes to zero

Wavelet transform and Hölder exponent estimation (Mallat, 2008)

Theorem (Mallat, 2008)

Let us suppose that the wavelet ψ has n vanishing moments. If $f \in L^2(\mathbb{R})$ is α-Hölder with $\alpha \leq n$ over $[a, b]$, then there exists $A > 0$ such that $\forall \alpha \in \mathbb{R}_+$

$$|Wf(x, s)| \leq As^{\alpha+1/2}, \;\; \forall x \in \mathbb{R}. \tag{1}$$

Conversely, suppose that f is bounded and that $Wf(x, s)$ satisfies (1) for an $\alpha < n$ that is not an integer. Then f is α-Hölder on $[a + \epsilon, b - \epsilon]$, for any $\epsilon > 0$.

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- When the scale s decreases, $Wf(x,s)$ measures fine scale variations in the local neighborhood of x: the inequality (1) is a condition on the asymptotic decay of $|Wf(x,s)|$ when s goes to zero
- At large scales it does not introduce any constraint since the Cauchy–Schwarz inequality guarantees that the wavelet transform is bounded

Wavelet transform and Hölder exponent estimation (Mallat, 2008)

- For fractal function on surfaces, formulation of wavelets on the corresponding discrete structure or in general in Riemannian manifolds
- Recent results use spectral representation of surfaces built upon their Laplace–Beltrami operator, with applications on brain surfaces from MRI [1]
- This requirement of a differential structure of the space make difficult to deal with metric space...

[1] H. Rabiei, O. Coulon, J. Lefèvre, F.J.P. Richard. Surface Regularity via the Estimation of Fractional Brownian Motion Index. IEEE Transactions on Image Processing, 30:1453–1460, 2021.

Morphological gradient and morphological total variation

- Morphological gradient (aka Beucher's gradient):

$$\beta(f)(x) = \lim_{\lambda \to 0} \frac{(f \oplus \lambda B)(x) - (f \ominus \lambda B)(x)}{2\lambda}$$

  where B is a closed unit disk in $\mathbb{R}^n$, and which equals $|\nabla f(x)|$ almost everywhere

- Discrete case of unit ball B:

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- Morphological gradient by structuring function $b(x)$:

$$\beta_b(f)(x) = (f \oplus b)(x) - (f \ominus b)(x).$$

Morphological gradient and morphological total variation

- Let Ω be an open subset of $\mathbb{R}^n$ and f a function belonging to $L^1(\Omega)$, its total variation is

$$TV(f) = \int_\Omega |\nabla f(x)|\, dx.$$

For a real-valued continuous function f, defined on an interval $[a, b] \subset \mathbb{R}$, its total variation is a measure of the one dimensional arclength of the curve $x \mapsto f(x)$:

$$TV(f) = \sup_{\mathcal{P}} \sum_{i=0}^{P-1} |f(x_{i+1}) - f(x_i)| \,,$$

where the supremum runs over the set of all partitions $\mathcal{P} = \{\ \pi = \{x_0, \cdots, x_P\} : \pi$ is a partition such that $x_0 = a$ and $x_p = b\ \}$

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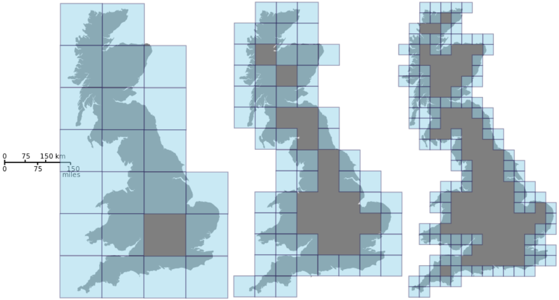
- Morphological total variation for any function $f : \mathbb{R}^2 \to \mathbb{R}$ w.r.t. a structuring fucntion b as

$$MTV_b(f) = \int_\Omega \beta_b(f)(x) dx = \int_\Omega [(f \oplus b)(x) - (f \ominus b)(x)]\, dx$$

Fractal dimensions and fractal sets

- Minkowski–Bouligand dimension (box-counting dimension)
  - Let us consider set $S \subset X$ is on an evenly spaced grid of metric space X
  - Count how many boxes are required to cover the set
  - The box-counting dimension is calculated by seeing how this number changes as we make the grid finer



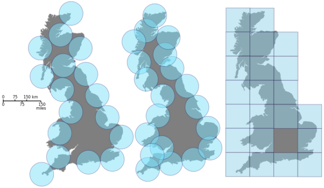
Source:
https://en.wikipedia.org/wiki/Minkowski_Bouligand_dimension

Fractal dimensions and fractal sets

- Minkowski–Bouligand dimension (box-counting dimension)
  - Suppose that N_ϵ is the number of boxes of side length ϵ required to cover the set
  - Then the box-counting dimension is defined as

$$\dim_{\mathrm{box}}(S) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}.$$

  - $N(\epsilon)$ displays an approximate power law with respect to the scale, i.e., $N(\epsilon) \sim \epsilon^{-\dim_{\mathrm{box}}(S)}$
  - Instead of boxes, the advantage of using balls B_r is that can be defined in any metric space



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Fractal dimensions and fractal sets

- Minkowski–Bouligand dimension (box-counting dimension)
  - In that case, the Minkowski–Bouligand dimension is given by

$$\dim_{\mathrm{M}}(S) = n - \lim_{\epsilon \to 0} \frac{\log \operatorname{vol}(S_\epsilon)}{\log \epsilon} = \lim_{\epsilon \to 0} \frac{\log [\operatorname{vol}(S_\epsilon)/\epsilon^n]}{\log [1/\epsilon]}$$

where for each radius $\epsilon > 0$, the set S_ϵ is defined to be the ϵ-neighborhood of S, i.e. the set of all points in R^n which are at distance less than ϵ from S

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  - Equivalently, S_ϵ, called the Minkowski cover, is the union of all the open balls B_ϵ of radius ϵ which are centered at a point in S, i.e.,

$$S_\epsilon = S \oplus B_\epsilon$$

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  - In the case of a compact set $S \subset \mathbb{R}^n$, we have

$$\dim_{\mathrm{box}}(S) = \dim_{\mathrm{M}}(S)$$

$$\dim_{\mathrm{H}}(S) \leq \dim_{\mathrm{M}}(S)$$

Fractal funtions

- A real valued function $f : \mathbb{R}^n \to \mathbb{R}$ is called fractal if its graph

$$\mathrm{Gr}(f) = ((x, a) \in \mathbb{R}^n \times \mathbb{R} \quad : \quad a = f(x))$$

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- If f is continuous, then its graph is a continuous curve with topological dimension equal to n:

$$f \text{ is continuous} \implies n \leq \dim_{\mathrm{H}}(\mathrm{Gr}(f)) \leq \dim_{\mathrm{M}}(\mathrm{Gr}(f)) \leq n+1$$

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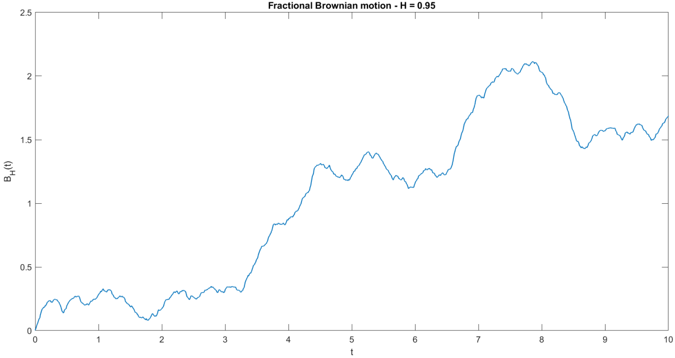
- Examples of fractal functions:
  - Weierstrass function
  - Fractional Brownian motion: Continuous-time Gaussian process $B_H(t)$ on $[0, T]$, that starts at zero, has expectation zero for all t in $[0, T]$, covariance function:

$$\mathbb{E}[B_H(t)B_H(s)] = \tfrac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

where $0 < H < 1$ is called the Hurst parameter

Fractional Brownian motion

- Sample-paths are nowhere differentiable and almost-all are Hölder continuous of any order strictly less than H
- The graph of $B_H(t)$ has $\dim_{\mathrm{H}} = \dim_{\mathrm{M}} = 2 - H$



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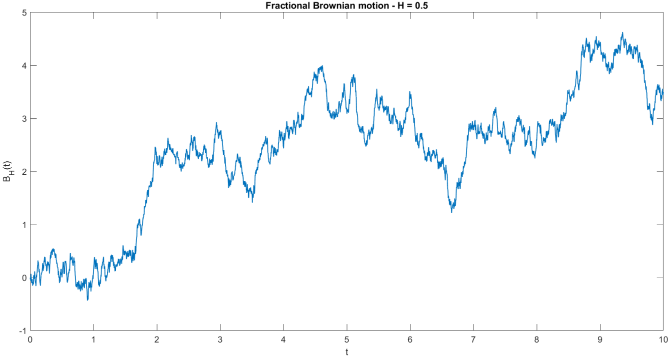

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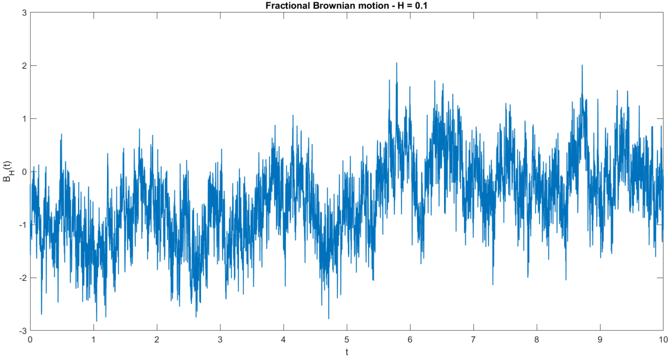

$\dim_M = 1.9$

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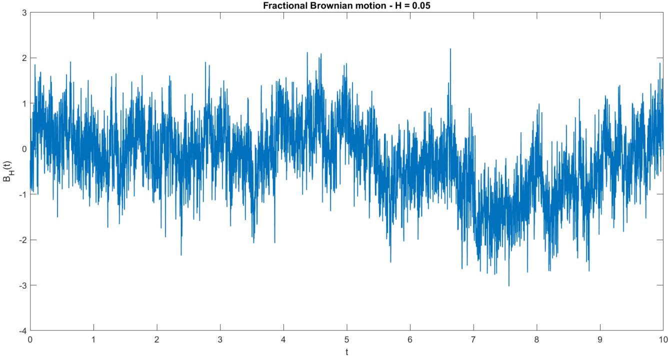
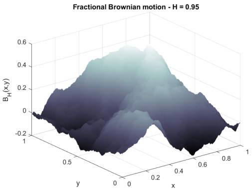


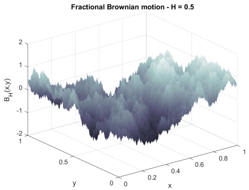
$\dim_{\mathrm{M}} = 1.95$

Fractional Brownian motion

2D case, $\dim_M = 3 - H$



$\dim_M = 2.05$



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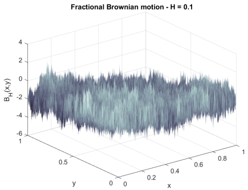

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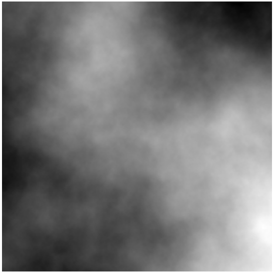

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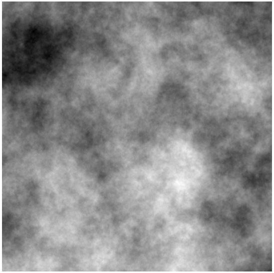

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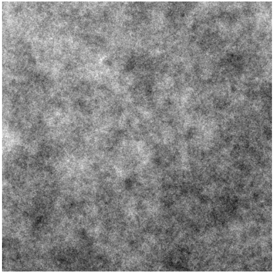

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Fractal analysis and Euclidean morphological operators

- Minkowski cover is just the dilation of of S by the homothetic ϵB of B at scale ϵ, which can be generalized to any structuring element B and define the notion of morphological cover by B:

$$C_\epsilon(S, B) = S \oplus \epsilon B$$

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- The covering blanket method extends this principle to function graphs: each point of the surface $\mathrm{Gr}(f)$ is replaced by a sphere B_ϵ of radius ϵ:

$$C_\epsilon(f, B) = \mathrm{Gr}(f) \oplus \epsilon B$$

Fractal analysis and Euclidean morphological operators

- In 2D: the area of the intensity surface is obtained by

$$\text{area}_\epsilon(f) = \frac{\text{vol}\left(\text{Gr}(f) \oplus B_\epsilon\right)}{\epsilon} = \epsilon^{-1} \int_{\mathbb{R}^2} \left[\left(f \oplus B_\epsilon\right)(x) - \left(f \ominus B_\epsilon\right)(x)\right] dx.$$

The volume of $\text{Gr}(f) \oplus B_\epsilon$ corresponds to the integral of the gradient of f by B_ϵ (Serra, 1982)

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- Using the notion of morphological total variation, one has:

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- Therefore for $f \in \mathcal{F}(\mathbb{R}^2, \mathbb{R})$, we have Therefore for $f \in \mathcal{F}(\mathbb{R}^2, \mathbb{R})$, we have

$$\begin{aligned} \dim\left(\text{Gr}(f)\right) &= 2 - \lim_{\epsilon \to 0} \frac{\log \text{area}_\epsilon\left(\text{Gr}(f)\right)}{\log \epsilon} \\ &= 3 - \lim_{\epsilon \to 0} \frac{\log \text{vol}\left(\text{Gr}(f) \oplus B_\epsilon\right)}{\log \epsilon} = \lim_{\epsilon \to 0} \frac{\log\left[MTV_{B_\epsilon}(f)/\epsilon^3\right]}{\log\left[1/\epsilon\right]}. \end{aligned}$$

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Fractal analysis and Euclidean morphological operators

- Instead of using $(n+1)$-dimensional structuring elements B: more consistent and computationaly efficient to work on functional morphological operators
- ϵ-scaled structuring function $b_\epsilon : \mathbb{R}^n \to \mathbb{R}$ for any homothetic ϵB of the structuring element $B \subseteq \mathbb{R}^{n+1}$:

$$b_\epsilon(x) = \sup \{a \ : \ (x,a) \in \epsilon B, \ x \in \mathbb{R}^n, \ a \in \mathbb{R}\}.$$

- Typical examples of symmetric structuring elements in the product space $\mathbb{R}^n \times \mathbb{R}$:

$$\epsilon B = \left\{(x,a) : \|x\|^2 + a^2 \leq \epsilon\right\} \quad \Longrightarrow \quad b_\epsilon(x) = \sqrt{\epsilon - \|x\|^2}, \ \ \|x\| \leq \epsilon, \ \ \text{(spherical)}$$

$$\epsilon B = \{(x,a) : \|x\|_1 + |a| \leq \epsilon\} \quad \Longrightarrow \quad b_\epsilon(x) = \epsilon - \|x\|_1, \ \ \|x\|_1 \leq \epsilon, \ \ (L_1 \text{ hat}).$$

Fractal analysis and Euclidean morphological operators

Theorem (Maragos and Sun, 1991)

Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Let $B \subseteq \mathbb{R}^{n+1}$ be a compact set, single connected and symmetric in the product space $\mathbb{R}^n \times \mathbb{R}$. We have

$$vol(C_\epsilon(f,B)) = \int_\Omega [(f \oplus b_\epsilon)(x) - (f \ominus b_\epsilon)(x)]\, dx.$$

Thus, one has: $vol(C_\epsilon(f,B)) = MTV_{b_\epsilon}(f)$, and therefore the fractal dimension of

$$\dim(Gr(f)) = (n+1) - \lim_{\epsilon \to 0} \frac{\log MTV_{b_\epsilon}(f)}{\log \epsilon}$$

Fractal analysis and Euclidean morphological operators

Use in practice

- The fractal dimension is obtained from by taking a structuring function b_ϵ, computing $MTV_{b_\epsilon}(f)$ for $\epsilon = 1, 2, ...$
- Then fitting a straight line using least squares of the graph $\log - \log$: the slope of this line give us $(n+1) - \dim(\mathrm{Gr}(f))$.
- For robust estimate: use various multi-scale structuring functions

Metric, length and geodesic space

- A metric space is a set of points X endowed with a distance function $d : X \times X \to [0, \infty)$. Let us assume that (X, d) is a complete separable metric space, locally compact (every closed ball or subset of X is compact)

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- A length space is a metric space (X, d) such that for any pair of points $x, y \in X$, we have

$$d(x, y) = \inf\{\mathrm{Length}(\sigma)\},$$

where the infimum is taken over all rectifiable curves $\sigma : [0, 1] \to X$ connecting x with y, i.e., $\sigma(0) = x$ and $\sigma(1) = y$

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- A curve σ is called a geodesic if σ has constant speed and if $\text{Length}(\sigma|_{[t,t']}) = d(\sigma(t), \sigma(t'))$, $\forall t, t' \in [0, 1]$, $t \leq t'$. A curve σ is a geodesic if for every two points $x, y \in X$, with $\sigma(0) = x$ and $\sigma(1) = y$, one has $d(\sigma(t), \sigma(t')) = |t - t'| d(x, y)$, $\forall t, t' \in [0, 1]$

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- (X, d) is a geodesic space if for every pair of points $x, y \in X$ there exists a geodesic $\sigma : [0, 1] \to X$ joining x to y
- Note that every geodesic space is a length space. Let X be a length space, complete and locally compact, then X is a geodesic space

Definition

- Assumptions. A metric space (X, d) and a given bounded function $f : X \mapsto \mathbb{R}$, being Lipschitz continuous. Let us consider a one-dimensional (shape) function $L : \mathbb{R}_+ \to \mathbb{R}_+$, being increasing, superlinear, convex of class $\mathcal{C}^1$ such that $L(0) = 0$

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- For all scales $t > 0$, the dilation $D_{L;\,t}f$ and the erosion $E_{L;\,t}f$ operators of f on (X, d) according to L are defined as

$$\begin{aligned} D_{L;\,t}f(x) &= \sup_{y \in X} \left\{ f(y) - tL\left(\frac{d(x,y)}{t}\right) \right\}, \quad \forall x \in X, \\ E_{L;\,t}f(x) &= \inf_{y \in X} \left\{ f(y) + tL\left(\frac{d(x,y)}{t}\right) \right\}, \quad \forall x \in X. \end{aligned}$$

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- Examples. A typical example of a shape function is $L(q) = q^P/P$, $P > 1$, such that

$$b_t(x-y) = -\frac{d(x,y)^P}{Pt^{P-1}}.$$

The canonic shape function corresponds to the case $P = 2$: $b_t(x-y) = -\frac{d(x,y)^2}{2t}$

Properties (1/2)

- (1) Adjunction. For any two real-valued functions f and g on (X,d), the pair $(E_{L;\,t}, D_{L;\,t})$ forms an adjunction, i.e., $D_{L;\,t}f(x) \leq g(x) \Leftrightarrow f(x) \leq E_{L;\,t}g(x), \quad \forall x \in X.$
- (2) Duality by involution. For any function f and $\forall x \in X$, one has

$$D_{L;\,t}f(x) = -E_{L;\,t}(-f)(x); \text{ and } E_{L;\,t}f(x) = -D_{L;\,t}(-f)(x), \ \forall t > 0.$$

- (3) Increaseness. If $f(x) \leq g(x)$, $\forall x \in X$, then

$$D_{L;\,t}f(x) \leq D_{L;\,t}g(x); \text{ and } E_{L;\,t}f(x) \leq E_{L;\,t}g(x), \ \forall x \in X, \forall t > 0.$$

- (4) Extensivity and anti-extensivity

$$D_{L;\,t}f(x) \geq f(x); \text{ and } E_{L;\,t}f(x) \leq f(x), \ \forall x \in X, \forall t > 0.$$

- (5) Ordering property If $0 < s < t$ then $\forall x \in X$

$$\inf_X f \leq E_{L;\,t}f(x) \leq E_{L;\,s}f(x) \leq f(x) \leq D_{L;\,s}f(x) \leq D_{L;\,t}f(x) \leq \sup_X f.$$

Properties (1/2)

- (6) Convergence For any function f and $\forall x \in X$, $D_{L;\,t}f(x)$ and $E_{L;\,t}f(x)$ converge monotonically to $f(x)$ as $t \to 0$. In particular $\lim_{t\to 0} D_{L;\,t}f = f$ and $\lim_{t\to 0} E_{L;\,t}f = f$.
- (7) Lipschitz The maps $(x,t) \mapsto D_{L;\,t}f(x)$ and $(x,t) \mapsto E_{L;\,t}f(x)$ are in $\mathrm{Lip}(X \times \mathbb{R}_+)$.
- (8) Semigroup For any function f and $\forall x \in X$, and for all pair of scales $s, t > 0$,
  - If X is metric space: $D_{L;\,t}D_{L;\,s}f \leq D_{L;\,t+s}f$; and $E_{L;\,t}E_{L;\,s}f \geq E_{L;\,t+s}f$.
  - X is a length space: $D_{L;\,t}D_{L;\,s}f = D_{L;\,t+s}f$; and $E_{L;\,t}E_{L;\,s}f = E_{L;\,t+s}f$.

PDE problem

- Let (X, d) be metric space. Let us consider the initial-value Hamilton–Jacobi first-order equation:

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) \pm H\left(|\nabla^{-} u(x,t)|\right) = 0, & \text{in } X \times (0,+\infty), \\ u(x,0) = f(x), & \text{in } X, \end{cases}$$

where the initial condition $f : X \to \mathbb{R}$ is a continuous bounded function and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is the Legendre transform of function $L(q)$:

$$H(p) = \max_{q \in \mathbb{R}_+} \{pq - L(q)\}, \quad p \in \mathbb{R}_+.$$

Theorem (Lott and Villani, 2007; Balogh et al.,2012)

The solutions of PDE problem are the dilation and erosion semigroups:

$$\begin{aligned} u(x,t) &= D_{L;\,t}f(x) \ \ (\text{for} - \text{sign}), \\ u(x,t) &= E_{L;\,t}f(x) \ \ (\text{for} + \text{sign}), \end{aligned}$$

in the following cases:

1. If (X,d) is a length space: solutions hold for all $x \in X$ and for almost everywhere $t > 0$.
2. If (X,d,μ) satisfies a doubling condition and supports a local Poincaré inequality: solutions hold for μ-almost everywhere $x \in X$ and for all $t > 0$.

Metric dilation and erosion semigroups of Hölder functions

Theorem (J.A, 2020)

(Part I) Let f be a real-valued function on the compact length space (X, d) with $f \in \bar{\mathcal{L}}_m$, where $m(d(x,y)) = Kd(x,y)^\alpha$, $0 < \alpha < 1$, $K > 0$. Let us consider the multi-scale structuring function

$$w_{P,t}(x,y) = -\frac{d(x,y)^P}{Pt^{P-1}},\ P > 1,\ \ t > 0$$

Then, the dilation $D_{P,t}(f)$ and erosion $E_{P,t}(f)$ belong to the class $f \in \bar{\mathcal{L}}_{m'}$ of Lipschitz functions, $m'(d(x,y)) = K_{P,t}d(x,y)$, i.e.,

$$\begin{aligned} |D_{P,t}(f)(x) - D_{P,t}(f)(y)| &\leq K_{P,t}d(x,y) \\ |E_{P,t}(f)(x) - E_{P,t}(f)(y)| &\leq K_{P,t}d(x,y) \end{aligned}$$

with constant $K_{P,t} = 2K^{\frac{P-1}{P-\alpha}} P^{\frac{\alpha-1}{P-\alpha}} t^{\frac{(1-P)^3}{P-\alpha}}$.

Metric dilation and erosion semigroups of Hölder functions

Theorem (J.A, 2020)

(Part II) In addition, one has the following bound on the variation

$$\begin{aligned}\sup_X |f - D_{P,t}(f)| &\leq K^{\frac{P}{P-\alpha}} P^{\frac{\alpha}{P-\alpha}} t^{\frac{(P-1)\alpha}{P-\alpha}},\\ \sup_X |f - E_{P,t}(f)| &\leq K^{\frac{P}{P-\alpha}} P^{\frac{\alpha}{P-\alpha}} t^{\frac{(P-1)\alpha}{P-\alpha}}.\end{aligned}$$

Metric dilation and erosion semigroups of Hölder functions

- This result of the Lipschitz regularity of dilation and erosion on metric spaces is just the counterpart of the classical ones for equicontinuous functions on Hilbert spaces, and they are the basic ingredients for the Lasry–Lions regularization, which can be also studied for Riemannian manifolds

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- Case of quadratic structuring function, i.e., $P = 2$, one gets:

$$
\begin{aligned}
|D_{2,t}(f)(x) - D_{2,t}(f)(y)| &\leq \left(2Kt^{-1}\right)^{\frac{1}{2-\alpha}} d(x,y), \\
\sup_X |f - D_{2,t}(f)| &\leq K^{\frac{2}{2-\alpha}} (2t)^{\frac{\alpha}{2-\alpha}} .
\end{aligned}
$$

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- These expressions provides a quantitative analysis of how Hölder functions are regularized by morphological semigroups on length spaces, but they cannot easily use to in practice to estimate the exponent α...

Hölder exponent estimates on metric spaces

Theorem (J.A, 2020)

Let (X, d) be a compact length space . For each power $P > 1$ and scale $t > 0$, we consider the multi-scale structuring function

$$w_{P,t}(x, y) = -\frac{d(x, y)^P}{Pt^{P-1}}.$$

and the corresponding metric dilation $D_{P,t}(f)$ and erosion $E_{P,t}(f)$ semigroups. The real-valued function on X, $f \in \mathcal{F}(X, \overline{\mathbb{R}})$ is α-Hölder, $0 < \alpha \leq 1$, if and only if it exists a constant C such as for $t > 0$, one has the following condition:

$$\beta_{b_{P,t}}(f)(x) \quad \leq \quad Ct^{\frac{(P-1)\alpha}{P-\alpha}},$$

with

$$C = 3K^{\frac{P}{P-\alpha}} P^{\frac{\alpha}{P-\alpha}},$$

where $\beta_{P,t}(f)(x) = D_{P,t}(f)(x) - E_{P,t}(f)(x)$.

Hölder exponent estimates on metric spaces

Use in practice

- Given a α-Hölder function, and by fixing $P > 0$, the expected value of the morphological gradient is:

$$\mathbb{E}\left[\beta_{P,t}(f)\right] \quad \leq \quad \mathbb{E}\left[Ct^{\frac{(P-1)\alpha}{P-\alpha}}\right]$$

- Using the corresponding morphological total variation: $MTV_{P,t}(f) = \int_X \beta_{P,t}(f)(x)d\mu(x)$, such that $\mathbb{E}\left[\beta_{P,t}(f)\right] = \mu(X)^{-1}MTV_{P,t}(f)$, with $\mu(X) = \int_X d\mu(x)$
- Thus one has near the origin ($t \to 0$):

$$\log MTV_{P,t}(f) = \log\left(C\mu(X)\right) + \left[\frac{(P-1)\alpha}{P-\alpha}\right] \log t,$$

and therefore, the slope s from the linear regression near the origin of the log-log curve $\log t \mapsto \log MTV_{P,t}(f) = a + s \log t$, provides the value of the Hölder exponent: $\alpha = Ps/((P-1)+s)$

Hölder exponent estimates on metric spaces

Experimental validation

- Fractional Brownian surfaces $B_H(x, y)$, $0 < H < 1$: a Hölder continuous function and the H provides an upper (tight) bound of the pointwise α-Hölder exponent
- For each experience, we have simulated 50 realizations of $B_H(x, y)$ with a given value of Hurst exponent H and we have extracted a fractal image field f of 500×500 pixels from each realisation
- We have considered five exponents: $H = 0.1$, $H = 0.3$, $H = 0.5$, $H = 0.7$ and $H = 0.9$

Hölder exponent estimates on metric spaces

Experimental validation

- Discretization of the shape function: Local neighbourhood for $w_{P,t}(x,y)$ of $L \times L$ pixels (outside, $w_{P,t}(x,y) = -\infty$), typically $L = 21$ or $L = 15$. Normalization of the distance:

$$d(x,y) = \frac{\|x-y\|_2}{L^2} \mathrm{rg}\, f$$

where $\mathrm{rg}\, f = \max f - \min f$ (typically, in our images, $\mathrm{rg} f \approx 3$)

- Semigroup property: To compute them using an iterative algorithm starting from the initial scale t_0. The idea for the dilation at the n-th scale is as follows:

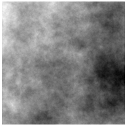
$$\begin{aligned} D_{P;\, t_0} f(x) &= \sup_{y\in X} \{f(y) + w_{P,t_0}(x,y)\} \\ &= \sup_{y\in X} \left\{ f(y) - \frac{d(x,y)^P}{Pt^{P-1}} \right\}, \end{aligned}$$

$$D_{P;\, t=nt_0} f(x) = D_{P;\, t_0} \left(D_{P;\, (n-1)t_0} f \right)(x), \quad n = 2,3,\cdots N.$$

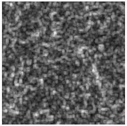
So finally, the important choice is the smallest scale $t = t_0$ and the number N of considered scales

Hölder exponent estimates on metric spaces

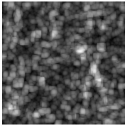
Experimental validation: Morphological metric gradient $\beta_{P,t=nt_0}(f)(x)$



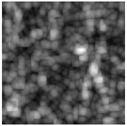
$f \ (H = 0.7)$



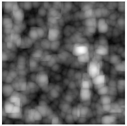
$t = t_0$



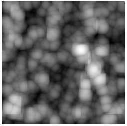
$t = 2t_0$



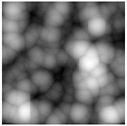
$t = 3t_0$



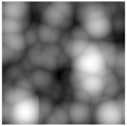
$t = 3t_0$



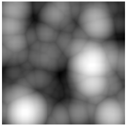
$t = 5t_0$



$t = 10t_0$



$t = 15t_0$

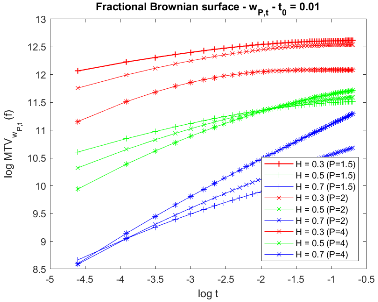


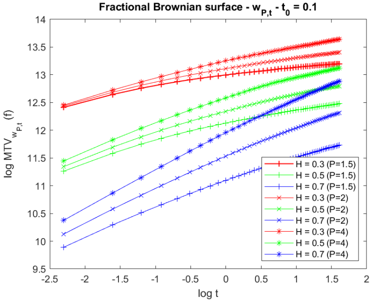
$t = 20t_0$

$(L = 15, \ P = 2$ and $t_0 = 0.1)$

Hölder exponent estimates on metric spaces

Experimental validation: Graphs in log – log scale of $(t, MTV_{P,t}(f))$





$(H = 0.3$ in red, 0.5 in green and 0.7 in blue$)$

Hölder exponent estimates on metric spaces

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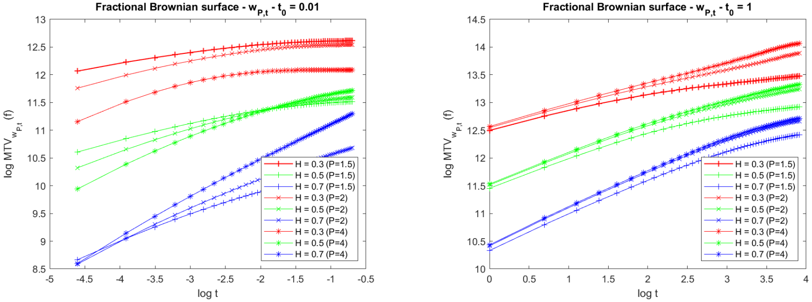

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Hölder exponent estimates on metric spaces

Experimental validation: Hölder exponent estimation α^*

	$w_{P,t}$, $t_0 = 0.01$ (15×15 neighbourhood)					
True	$P = 1.5$		$P = 2$		$P = 4$	
H	Mean ± StdDev	Error (%)	Mean ± StdDev	Error (%)	Mean ± StdDev	Error (%)
0.1	0.15 ± 0.003	53	0.14 ± 0.004	39	0.14 ± 0.004	39
0.3	0.29 ± 0.007	1.08	0.29 ± 0.01	4.2	0.21 ± 0.04	30
0.5	0.45 ± 0.02	9.6	0.46 ± 0.03	7.0	0.49 ± 0.05	0.5
0.7	0.62 ± 0.02	11.4	0.64 ± 0.02	7.7	0.69 ± 0.03	1.5
0.9	0.79 ± 0.02	12	0.81 ± 0.04	10.2	0.81 ± 0.05	9.9

	$w_{P,t}$, $t_0 = 0.1$ (15×15 neighbourhood)					
True	$P = 1.5$		$P = 2$		$P = 4$	
H	Mean ± StdDev	Error (%)	Mean ± StdDev	Error (%)	Mean ± StdDev	Error (%)
0.1	0.21 ± 0.008	114	0.22 ± 0.007	116	0.22 ± 0.008	124
0.3	0.38 ± 0.02	25.6	0.36 ± 0.02	21.3	0.35 ± 0.03	15.6
0.5	0.53 ± 0.02	5.8	0.51 ± 0.02	2.5	0.48 ± 0.03	3.3
0.7	0.69 ± 0.05	1.9	0.67 ± 0.05	7.7	0.62 ± 0.05	10.6
0.9	0.83 ± 0.03	7.6	0.81 ± 0.04	9.3	0.77 ± 0.05	14

Conclusions

- Morphological operators (or max-plus wavelets), for the estimation of the Minkoswki dimension from fractal functions or the exponent from Hölder functions is a classic topic for functions on $\mathbb{R}^n$

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Conclusions

- Morphological operators (or max-plus wavelets), for the estimation of the Minkoswki dimension from fractal functions or the exponent from Hölder functions is a classic topic for functions on $\mathbb{R}^n$
- Morphology operators can be extended to functions on length spaces, including the corresponding Hamilton–Jacobi partial differential equations and their solutions as morphological semigroups
- We have shown that Hamilton–Jacobi semigroups on length spaces are the main ingredients to characterize Hölder functions on that rather general space family
- Euclidean and Riemannian manifolds belong to this class, as well as other discrete geodesic spaces such as networks and graphs

Perspectives

- This theory provides an alternative approach to wavelets as tool to characterize Hölder functions:
  - Morphological semigroups are more naturally extended to non-Euclidean spaces than wavelets
  - In the case of high dimensional vector spaces, morphological semigroups are efficiently computed, i.e., the basic ingredient is just the distance between points

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  - Morphological semigroups are more naturally extended to non-Euclidean spaces than wavelets
  - In the case of high dimensional vector spaces, morphological semigroups are efficiently computed, i.e., the basic ingredient is just the distance between points
- Two main potential applications of this theory in the field of image and data analysis:
  - Morphological sampling of real-valued functions on high dimensional spaces preserving Hölder continuity
  - Formulation of ad-hoc architectures of NN adapted to predict (classification and regression) fractal dimension and similar underlaying regularity parameters of functions such as textures, sounds or other physical signals