Characterization of Insertable and Removable Pixels for Digital Convex Sets

Lama Tarsissi,
David Coeurjolly, Yukiko Kenmochi, Pascal Romon and Jean-Pierre Borel

Journées de Géométrie Discrète et Morphologie Mathématique
Nancy, 16-03-2021

When we talk about inflation or deflation!!!

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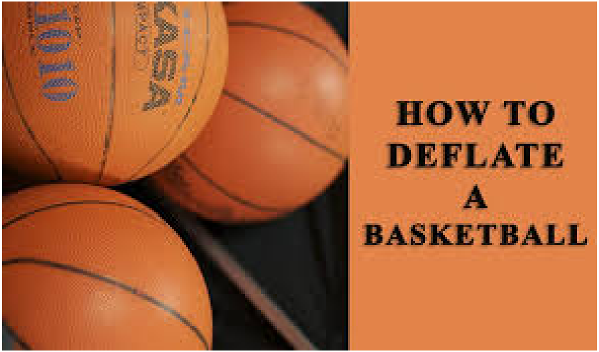

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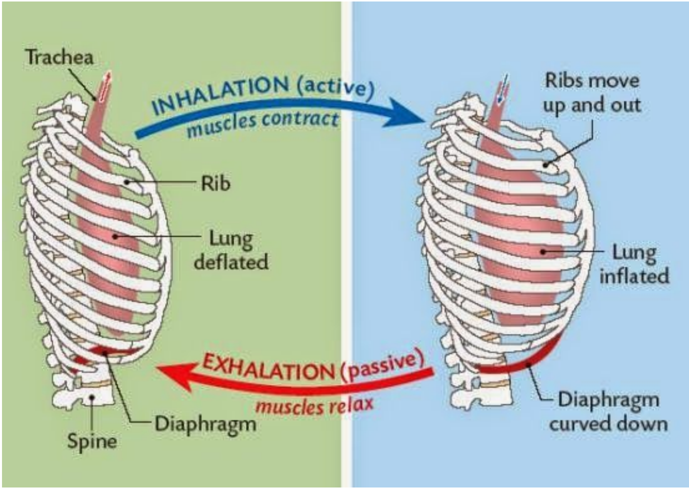

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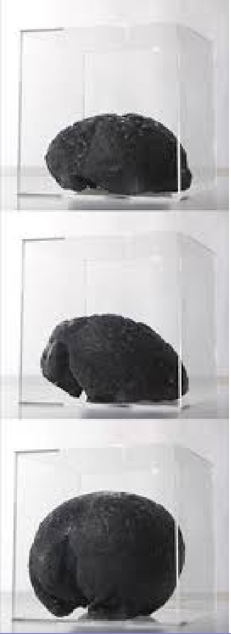

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Outlines

1. Digital convex Sets in Combinatorics on Words point of view
2. Characterizations for removable pixels by preserving the digital convexity
3. Characterizations for insertable pixels by preserving the digital convexity
4. The utility of Combinatorics on words for the inflating process

Digital convexity-DC

Convexity

In $\mathbb{R}^2$, a subset R is said to be convex if for any pair of points $x, y \in R$, every point on the straight line segment joining x and y is also within R.

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Under the connectivity assumption, Ekhdart has shown that the different definitions coincide.

In this presentation, we consider finite and 4-connected sets of $\mathbb{Z}^2$.

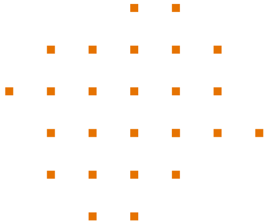
And we use the definition of digital convexity based on convex hull as follows:

Definition

A finite 4-connected set S of $\mathbb{Z}^2$ is *digitally convex* if $Conv(S) \cap \mathbb{Z}^2 = S$.

Terminology

- $C \subset \mathbb{Z}^2$, finite 4-connected digital convex set.



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- $C \subset \mathbb{Z}^2$, finite 4-connected digital convex set.
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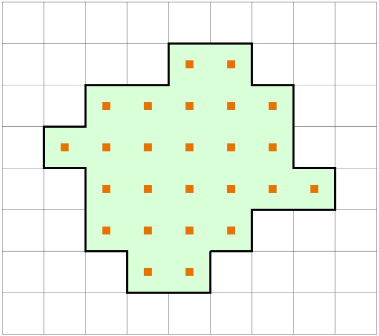

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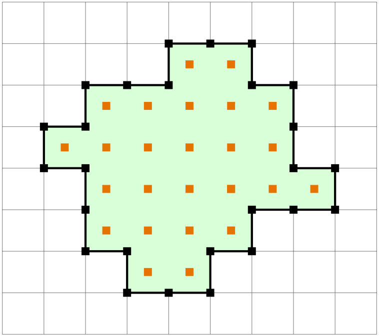
- $Bd(V(C))$ the topological boundary of $V(C)$; Bd(C)



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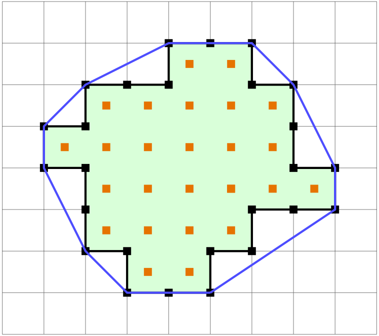
- $Bd(V(C))$ the topological boundary of $V(C)$; Bd(C)
- Convex hull of Bd(C), denoted by $conv(Bd(C))$



Main questions

1. For a finite, 4-connected and digitally convex set C of $\mathbb{Z}^2$ and a point x of C, we would like to verify if $C \setminus \{x\}$ (resp. $C \cup \{x\}$) is still digitally convex (and 4-connected)

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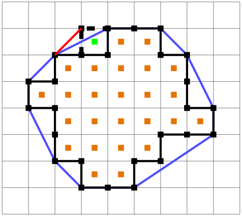

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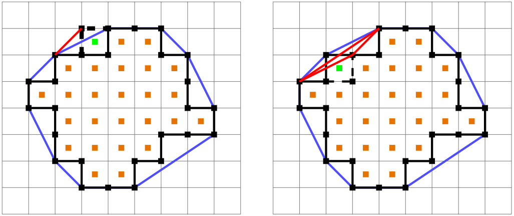

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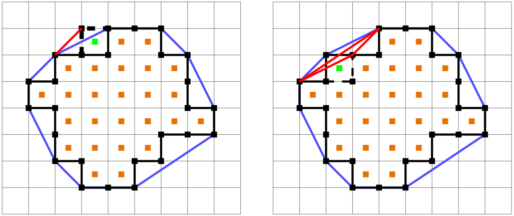

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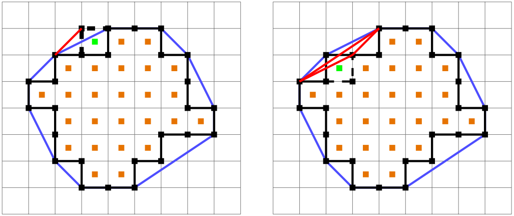

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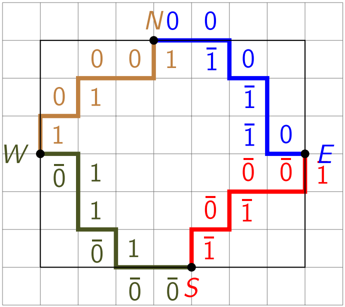
Our approach to solve these questions is based on Combinatorics on words by studying the boundary word of C, W(C).

Boundary word



The boundary word of DC is $W(C) = 10100100\bar{1}0\bar{1}\bar{1}01\overline{0010100}1\bar{0}11\bar{0}$.

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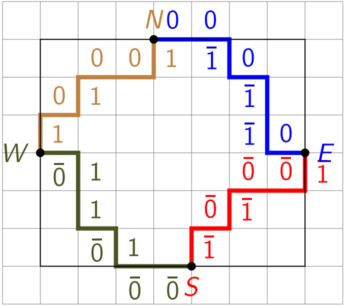

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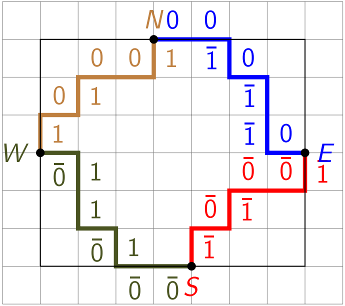

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$$W(C) = (1)(01)(001)0^2(\bar{1}0)(\overline{11}0)(1)(\overline{00101})(\bar{0})^2(1\bar{0})(11\bar{0})$$

where each factor is a Christoffel word.

Theorem (BLPR09)

A word $w \in \{0,1\}^*$ is WN-convex if and only if its Lyndon factorization is unique, $w_1^{n_1} w_2^{n_2} \dots w_k^{n_k}$, and their factors are all primitive Christoffel words.

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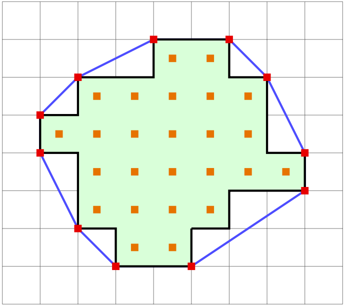
Property

Given a 4-connected digitally convex set C, each vertex of the $conv(Bd(C))$, corresponds to the end of each factor of the Lyndon factorization of $W(C)$.



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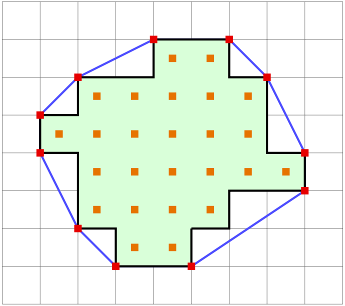
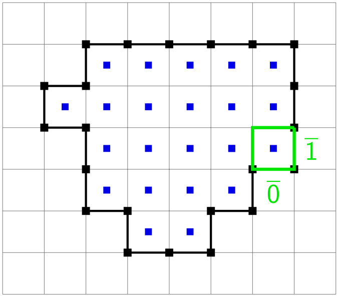


The red points over $W(C)$ correspond to the Lyndon pixels of $V(C)$.

Deflation



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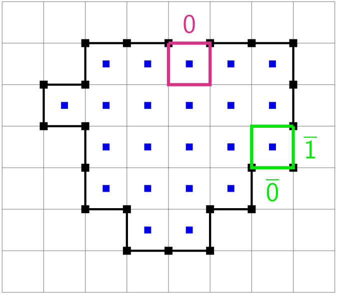

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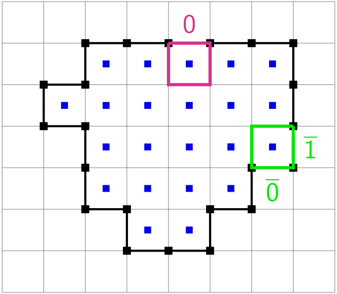

Theorem

Given a 4-connected, digitally convex set C of $\mathbb{Z}^2$, let us consider its boundary word $W(C)$ and its Lyndon factorization given by $W(C) = \ell_1^{n_1} \ell_2^{n_2} \dots \ell_s^{n_s}$. A pixel $p(x)$ for a certain $x \in C$ is removable if x is a simple point and $p(x)$ is a Lyndon pixel.

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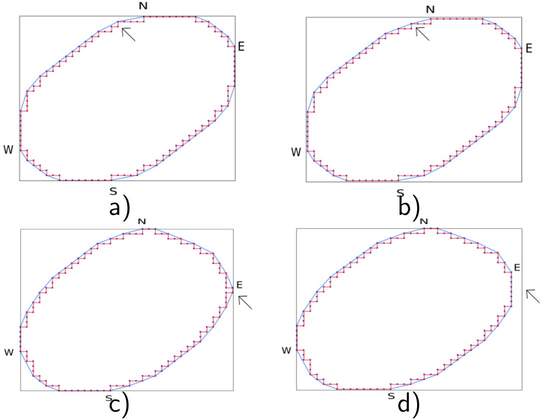

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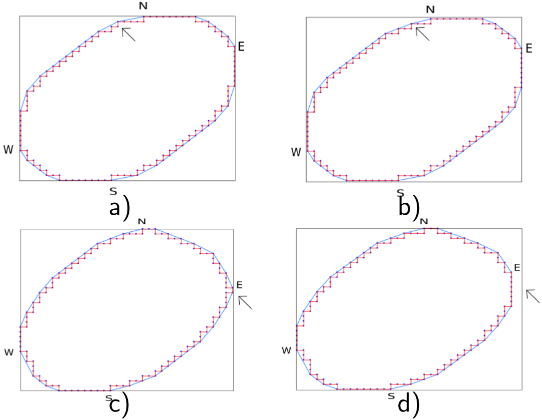
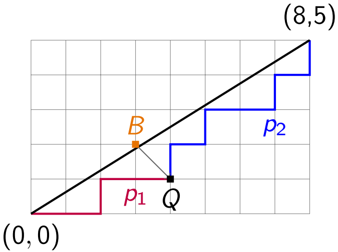


4. Avoid the choice of the pixel that will lead to a $W(C)$ of the form $1^k0^l\overline{1}^k\overline{0}^l$

Furthest point



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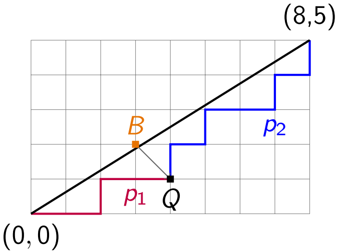

$Q = fu(w)$, over the Christoffel word of slope $\frac{5}{8}$.
Its corresponding pixel is called the furthest pixel .

Switching position and local modifications for the inflation

Lemma(Tarsissi et al-17)

Let $w = u.v$ be a Christoffel word of length strictly greater than 1, with u and v the two Christoffel factors. Switching letters at the furthest point position k of w, switches the places of the Christoffel words u and v, and we get:

$$switch_k(w) = (w^+, w^-) = v.u; \quad where\ switch_k(w) = vu,\ and\ \rho(v) > \rho(u).$$

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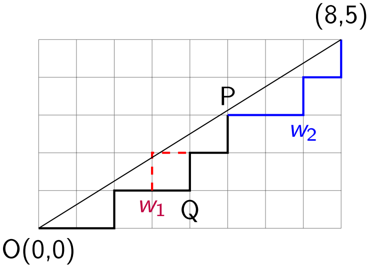

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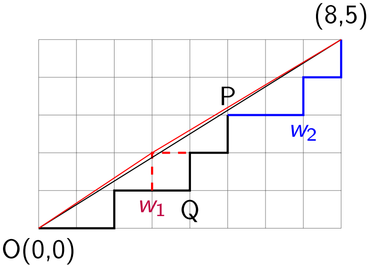

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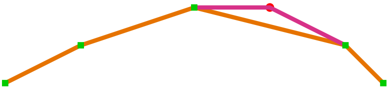

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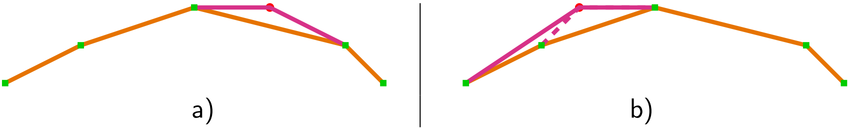

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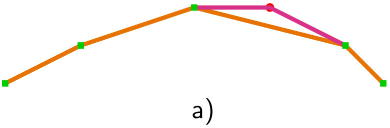

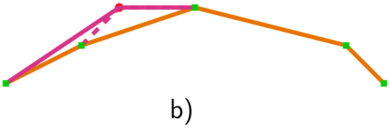

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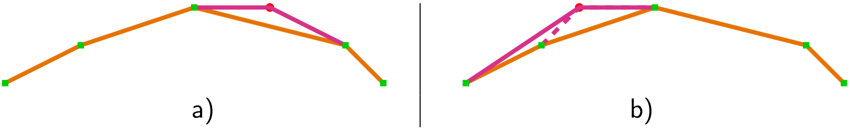

OR Loosing convexity

Let $w_1 = C(\frac{30}{41})$ and $w_2 = C(\frac{5}{7})$,

$$switch_k(w_1)w_2 = C(\frac{11}{15})C(\frac{19}{26})C(\frac{5}{7}); \; w_1 switch_{k'}(w_2) = C(\frac{30}{41})C(\frac{3}{4})C(\frac{2}{3}).$$

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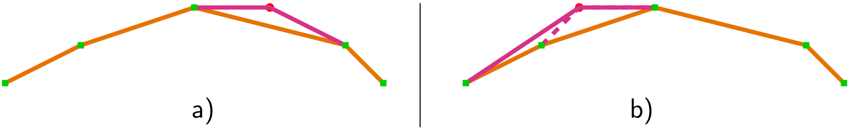

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$\frac{11}{15} > \frac{19}{26} > \frac{5}{7}$, while $\frac{30}{41} < \frac{3}{4}$.

Global Digital convexity verification and its algorithm

Theorem

Let $W(C) = \ell_1^{n_1} \ldots \ell_s^{n_s}$ be a boundary word of a 4-connected digital convex set C. By switching two letters of the first Christoffel word ℓ_1 at the furthest point position , we obtain two line segments: V_0V discretized by ℓ_1^+ and VV_1 discretized by $\ell_1^- \ell_1^{n_1-1}$.

1. *If $\ell_2 < \ell_1^- \ell_1^{n_1-1}$,*
2. *If $\ell_2 = \ell_1^- \ell_1^{n_1-1}$; i.e. ℓ_2 is aligned with $\ell_1^- \ell_1^{n_1-1}$,*
3. *If $\ell_2 = (\ell_1^- \ell_1^{n_1-1})^{m_1} \ell_1$, with $m_1 \geq 1$, then we check the propagation by concatenating $\ell_1^- \ell_1^{n_1-1}$ and ℓ_2,*
4. *Otherwise, we loose the convexity and this point should not be chosen.*

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Being a furthest pixel is a sufficient condition for the insertable pixel.

Theorem:Strong condition to characterize an insertable pixel

Let $W(C) = \ell_1^{n_1} \ldots \ell_s^{n_s}$ and ℓ_j be a primitive Christoffel word of maximal length. Applying the $switch_k(\ell_j) = (\ell_j^+, \ell_j^-)$ then the new Lyndon factorization gives:

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1. If $(i > 1 \text{ } or \text{ } \ell_{j-1} > \ell_j^+)$ and $(i < n_j \text{ } or \text{ } \ell_{j+1} < \ell_j^-)$:

$$\ell_1^{n_1} \ell_2^{n_2} \dots \ell_{j-1}^{n_{j-1}} (\ell_j^{i-1} \ell_j^+)(\ell_j^- \ell_j^{n_j - i}) \ell_{j+1}^{n_{j+1}} \dots \ell_k^{n_k} .$$

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2. If ($i = 1$ *and* $\ell_{j-1} = \ell_j^+$) and ($i < n_j$ *or* $\ell_{j+1} < \ell_j^-$):

$$\ell_1^{n_1}\, \ell_2^{n_2} \dots \ell_{j-1}^{n_{j-1}+1} (\ell_j^-\, \ell_j^{n_j - i})\, \ell_{j+1}^{n_{j+1}} \dots \ell_k^{n_k}\, .$$

3. If ($i > 1$ *or* $\ell_{j-1} > \ell_j^+$) and ($i = n_j$ *or* $\ell_{j+1} = \ell_j^-$):

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Theorem:Strong condition to characterize an insertable pixel

Let $W(C) = \ell_1^{n_1} \dots \ell_s^{n_s}$ and ℓ_j be a primitive Christoffel word of maximal length. Applying the $switch_k(\ell_j) = (\ell_j^+, \ell_j^-)$ then the new Lyndon factorization gives:

1. If $(i > 1 \; or \; \ell_{j-1} > \ell_j^+)$ and $(i < n_j \; or \; \ell_{j+1} < \ell_j^-)$:

$$\ell_1^{n_1}\, \ell_2^{n_2} \dots \ell_{j-1}^{n_{j-1}} (\ell_j^{i-1}\, \ell_j^+)(\ell_j^-\, \ell_j^{n_j - i})\, \ell_{j+1}^{n_{j+1}} \dots \ell_k^{n_k} .$$

2. If $(i = 1 \; and \; \ell_{j-1} = \ell_j^+)$ and $(i < n_j \; or \; \ell_{j+1} < \ell_j^-)$:

$$\ell_1^{n_1}\, \ell_2^{n_2} \dots \ell_{j-1}^{n_{j-1}+1} (\ell_j^-\, \ell_j^{n_j - i})\, \ell_{j+1}^{n_{j+1}} \dots \ell_k^{n_k} .$$

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$$\ell_1^{n_1}\, \ell_2^{n_2} \dots \ell_{j-1}^{n_{j-1}} (\ell_j^{i-1}\, \ell_j^+)\, \ell_{j+1}^{n_{j+1}+1} \dots \ell_k^{n_k} .$$

4. If $(i = 1 \; and \; \ell_{j-1} = \ell_j^+)$ and $(i = n_j \; and \; \ell_{j+1} = \ell_j^-)$:

$$\ell_1^{n_1}\, \ell_2^{n_2} \dots \ell_{j-1}^{n_{j-1}+1}\, \ell_{j+1}^{n_{j+1}+1} \dots \ell_k^{n_k} .$$

Sketch proof

The proof of this theorem relies on two points:

1. Showing that $\ell_j^{i-1}\,\ell_j^+$ and $\ell_j^-\,\ell_j^{n_j-i}$ are Christoffel words.
2. Proving the following inequalities: $\ell_{j-1} > \ell_j^{i-1}\,\ell_j^+ > \ell_j^-\,\ell_j^{n_j-i} > \ell_{j+1}$.
   - The inequality in the middle by applying some Christoffel morphisms.
   - If the last inequality is not correct, we have: $\ell_j^- \leq \ell_j^-\,\ell_j^{n_j-i} \leq \ell_{j+1} < \ell_j$. Then ℓ_{j+1} is a Christoffel word in the angle of ℓ_j^-, ℓ_j and can't be equal to ℓ_j, in this case it has to be longer than ℓ_j, contradiction to the main condition that ℓ_j is the longest Christoffel.
   - The first inequality is treated in a symmetric way as the previous one.

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1. The propagation doesn't exceed the next Christoffel word $\ell_{j+1}^{n_j+1}$ or the previous one $\ell_{j-1}^{n_j-1}$.

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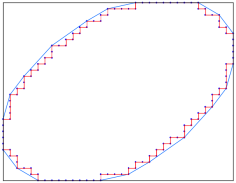

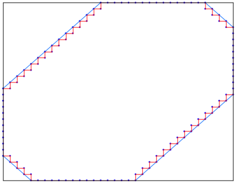

Some Perspectives

1. The algorithmic details and optimization of the process.
2. The choice of the optimal heuristic for deflating a digital convex set.
3. Apply these algorithms on non-convex shapes by studying the locally convex boundary using combinatorics on words.

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THANK YOU