An alternative definition for digital convexity

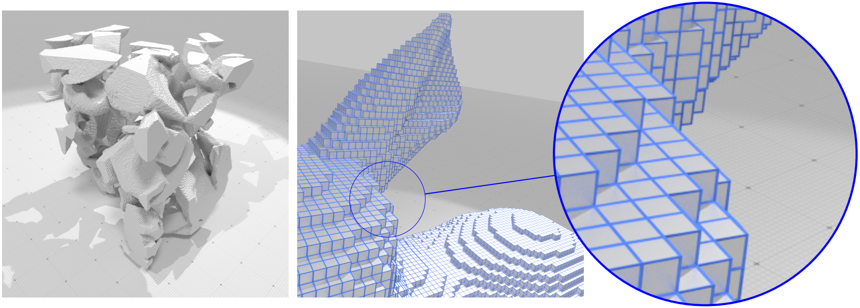
Jacques-Olivier Lachaud[1]

[1]Laboratory of Mathematics
University Savoie Mont Blanc

March 16th, 2021
Journée du GT Géométrie Discrète et Morphologie Mathématique
LORIA, Nancy

An alternative definition for digital convexity

Why digital convexity ?

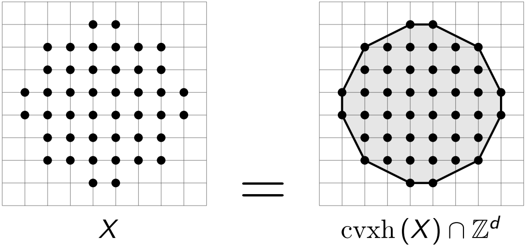


- no (infinitesimal) differential geometry for digital shapes
- convexity: a fundamental tool to analyze the geometry of shapes
- identifies convex/concave/flat/saddle regions
- gives locally its piecewise linear geometry
- facets give normal estimations

Natural digital convexity is not satisfactory

Definition (Natural digital convexity (or *H*-convexity))

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{cvxh}\,(X) \cap \mathbb{Z}^d = X$

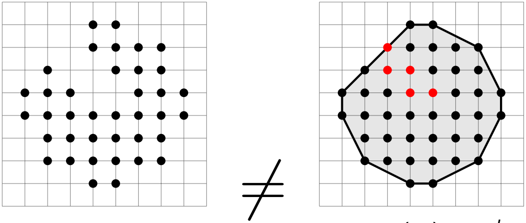


$\Rightarrow$ convex

Natural digital convexity is not satisfactory

Definition (Natural digital convexity (or *H*-convexity))

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{cvxh}\,(X) \cap \mathbb{Z}^d = X$

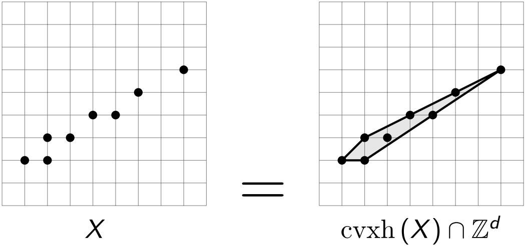


X $\neq$ $\mathrm{cvxh}\,(X) \cap \mathbb{Z}^d$ $\Rightarrow$ not convex

Natural digital convexity is not satisfactory

Definition (Natural digital convexity (or *H*-convexity))

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{cvxh}\,(X) \cap \mathbb{Z}^d = X$

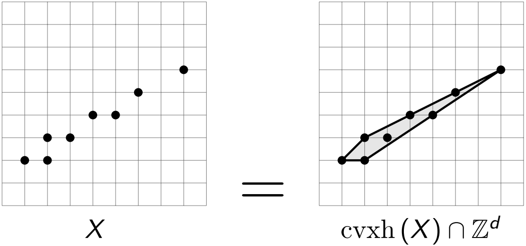


$\Rightarrow$ convex !

Natural digital convexity is not satisfactory

Definition (Natural digital convexity (or *H*-convexity))

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{cvxh}(X) \cap \mathbb{Z}^d = X$



$\Rightarrow$ convex !

Digital convexity does not imply digital connectedness !

Usual digital convexity adds connectedness

Definition (Usual digital convexity)

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{cvxh}(X) \cap \mathbb{Z}^d = X$ **and** X connected

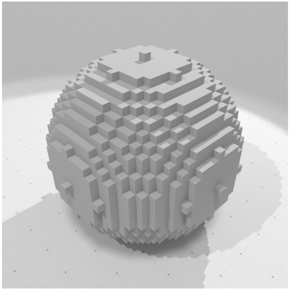
- many more or less equivalent definitions **in 2D**: straight segment convexity, triangle convexity, . . . [Minsky, Papert 88], [Kim, Rosenfeld 83], [Huübler, Klette, Voss89], . . .

Usual digital convexity adds connectedness

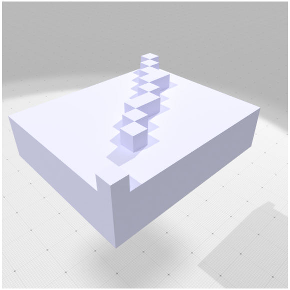
Definition (Usual digital convexity)

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{cvxh}(X) \cap \mathbb{Z}^d = X$ **and** X connected

- many more or less equivalent definitions **in 2D**: straight segment convexity, triangle convexity, ... [Minsky, Papert 88], [Kim, Rosenfeld 83], [Huübler, Klette, Voss89], ...
- **none extends well to 3D or more**



convex



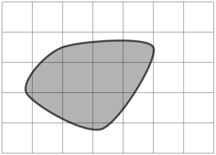
convex !

An alternative definition for digital convexity

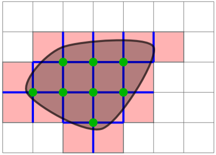
Cubical grid, intersection complex

- cubical grid complex $\mathcal{C}^d$
  - $\mathcal{C}_0^d$ vertices or 0-cells $= \mathbb{Z}^d$
  - $\mathcal{C}_1^d$ edges or 1-cells $=$ open unit segment joining 0-cells
  - $\mathcal{C}_2^d$ faces or 2-cells $=$ open unit square joining 1-cells
  - ...
- *intersection complex* of $Y \subset \mathbb{R}^d$

$$\bar{\mathcal{C}}_k^d[Y] := \{c \in \mathcal{C}_k^d, \bar{c} \cap Y \neq \emptyset\}$$



Y



cells $\bar{\mathcal{C}}_0^d[Y]$, $\bar{\mathcal{C}}_1^d[Y]$, $\bar{\mathcal{C}}_2^d[Y]$

Full convexity

Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}_k^d[X] = \bar{\mathcal{C}}_k^d[\mathrm{cvxh}\,(X)]. \tag{1}$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.

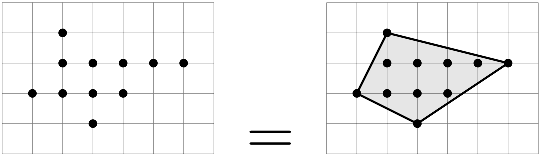
Full convexity

Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}_k^d[X] = \bar{\mathcal{C}}_k^d[\mathrm{cvxh}\,(X)]. \tag{1}$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.



$\bar{\mathcal{C}}_0^d[X]$ = $\bar{\mathcal{C}}_0^d[\mathrm{cvxh}\,(X)]$

X is digitally 0-convex

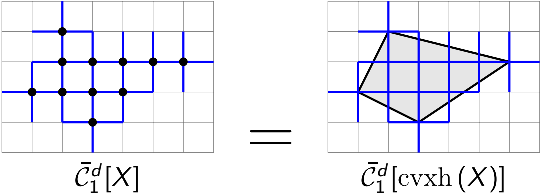
Full convexity

Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}_k^d[X] = \bar{\mathcal{C}}_k^d[\mathrm{cvxh}\,(X)]. \qquad (1)$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.



X is digitally 0-convex, and 1-convex

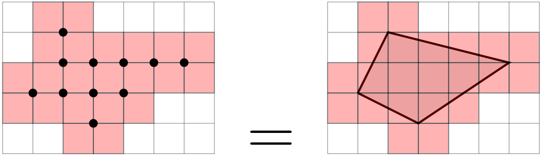
Full convexity

Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}_k^d[X] = \bar{\mathcal{C}}_k^d[\mathrm{cvxh}\,(X)]. \tag{1}$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.



$\bar{\mathcal{C}}_2^d[X]$ $=$ $\bar{\mathcal{C}}_2^d[\mathrm{cvxh}\,(X)]$

X is digitally 0-convex, and 1-convex, and 2-convex, hence fully convex.

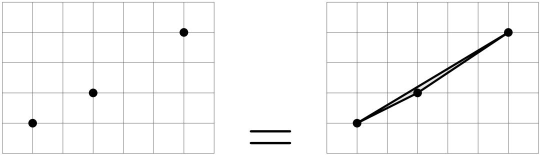
Full convexity

Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}_k^d[X] = \bar{\mathcal{C}}_k^d[\mathrm{cvxh}\,(X)]. \tag{1}$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.



$\bar{\mathcal{C}}_0^d[X]$ = $\bar{\mathcal{C}}_0^d[\mathrm{cvxh}\,(X)]$

X is digitally 0-convex

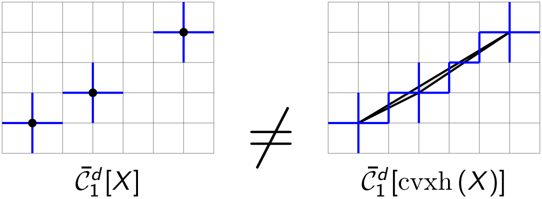
Full convexity

Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}_k^d[X] = \bar{\mathcal{C}}_k^d[\mathrm{cvxh}\,(X)]. \quad (1)$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.



X is digitally 0-convex, but neither 1-convex

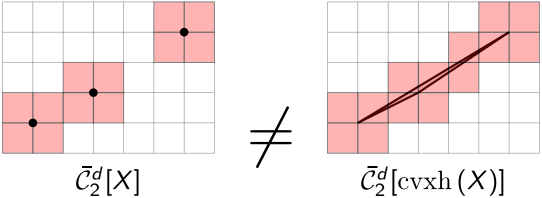
Full convexity

Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}_k^d[X] = \bar{\mathcal{C}}_k^d[\mathrm{cvxh}\,(X)]. \tag{1}$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.



X is digitally 0-convex, but neither 1-convex, nor 2-convex.

Full convexity

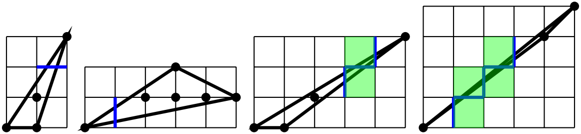
Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}_k^d[X] = \bar{\mathcal{C}}_k^d[\mathrm{cvxh}\,(X)]. \quad (1)$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.

Full convexity eliminates too thin digital convex sets in arbitrary dimension.



An alternative definition for digital convexity

Digital connectedness

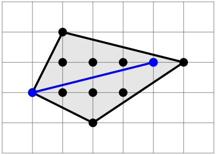
Theorem

If the digital set $X \subset \mathbb{Z}^d$ is fully convex, then X is d-connected.

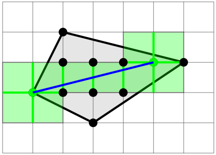
Proof.

- for $x, y \in X$, segment $x - y$ intersects cells $c_0, c_1, \ldots, c_m$,
- each c_i touches at least one corner $z_i \in X$,
- each c_i is a face of c_{i+1} or inversely,
- implies z_i and z_{i+1} shares a unit cube, hence d-connected

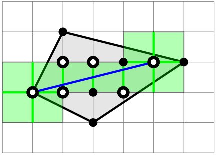
□



$[x, y]$



intersected cells c_i



points z_i

Simple connectedness

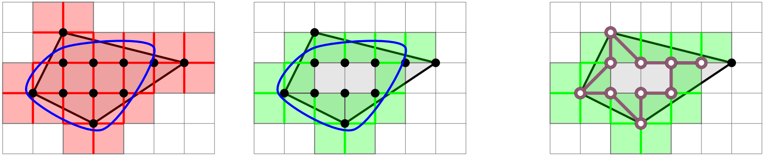
Theorem

If the digital set $X \subset \mathbb{Z}^d$ is fully convex, then the body of its intersection complex is **simply** *connected.*

Proof.

- let $\mathcal{A} := \{x(t), t \in [0,1]\}$ be a closed curve in $\left\|\bar{\mathcal{C}}^d[X]\right\|$
- sequence of intersected cells $c_i \in \bar{\mathcal{C}}^d[X]$
- sequence of associated corners $z_i \in X$
- homotopy between $\mathcal{A}$ and path $z_0 - z_1 - \cdots - z_n - z_0$
- path $z_0 - z_1 - \cdots - z_n - z_0$ subset of $\mathrm{cvxh}\,(X) \Rightarrow$ contractible

$\square$

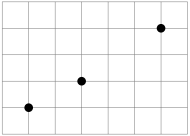


$\mathcal{A}$ — intersected cells (c_i) — path $z_0 - z_1 - \cdots - z_n - z_0$

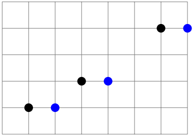
An alternative definition for digital convexity

Discrete Minkowski sum U_α

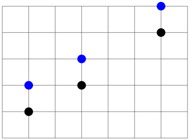
- let $X \subset \mathbb{Z}^d$, denote $\mathrm{e}_i(X)$ the translation of X with axis vector e_i
- let $I^d := \{1, \dots, d\}$ be the set of possible directions
- let $U_\emptyset(X) := X$, and, for $\alpha \subset I^d$ and $i \in \alpha$, recursively $U_\alpha(X) := U_{\alpha \setminus i}(X) \cup \mathrm{e}_i(U_{\alpha \setminus i}(X))$.



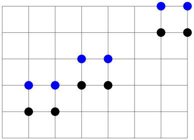
$U_\emptyset(X) = X$



$U_{\{1\}}(X) = U_\emptyset(X) \cup \mathrm{e}_1(U_\emptyset(X))$



$U_{\{2\}}(X) = U_\emptyset(X) \cup \mathrm{e}_2(U_\emptyset(X))$



$U_{\{1,2\}}(X) = U_{\{1\}}(X) \cup \mathrm{e}_1(U_{\{1\}}(X))$

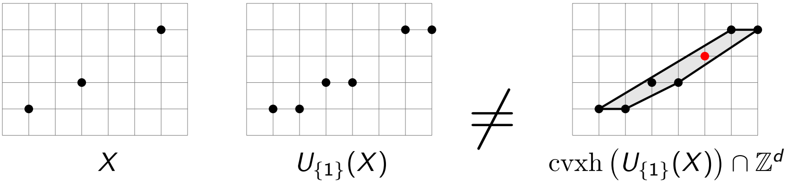
A morphological characterization

Theorem

A non empty subset $X \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k \leqslant d$ iff

$$\forall \alpha \in I_k^d, U_\alpha(X) = \mathrm{cvxh}\left(U_\alpha(X)\right) \cap \mathbb{Z}^d. \tag{2}$$

It is thus fully convex if the previous relations holds for all $k, 0 \leqslant k \leqslant d$.



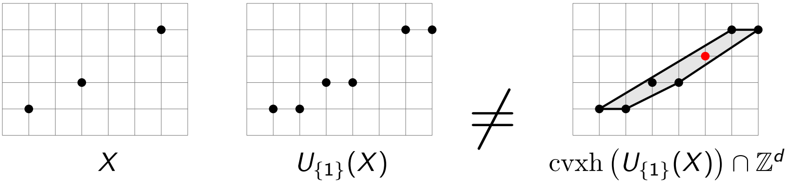
A morphological characterization

Theorem

A non empty subset $X \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k \leqslant d$ iff

$$\forall \alpha \in I_k^d, U_\alpha(X) = \mathrm{cvxh}\left(U_\alpha(X)\right) \cap \mathbb{Z}^d. \tag{2}$$

It is thus fully convex if the previous relations holds for all $k, 0 \leqslant k \leqslant d$.

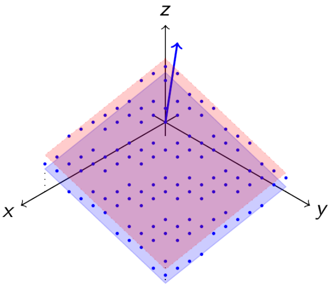


Algorithm in arbitrary dimension

$U_\alpha(X)$ easily computed while convex hull algorithms exist in arbitrary dimension. Slowest part is lattice point enumeration in convex hull.

An alternative definition for digital convexity

Thick enough arithmetic planes are full convex

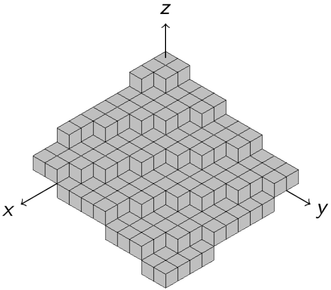


Arithmetic plane

- irreducible normal vector $N \in \mathbb{Z}^d$
- intercept $\mu \in \mathbb{Z}$
- positive thickness $\omega \in \mathbb{Z}, \omega > 0$

$$P(\mu, N, \omega) := \{x \in \mathbb{Z}^d, \mu \leqslant x \cdot N < \mu + \omega\}$$

Thick enough arithmetic planes are full convex

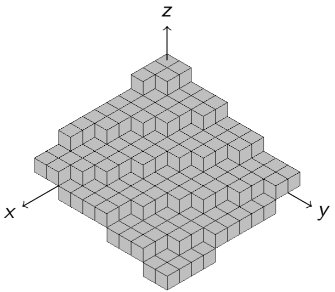


Arithmetic plane

- irreducible normal vector $N \in \mathbb{Z}^d$
- intercept $\mu \in \mathbb{Z}$
- positive thickness $\omega \in \mathbb{Z}, \omega > 0$

$$P(\mu, N, \omega) := \{x \in \mathbb{Z}^d, \mu \leqslant x \cdot N < \mu + \omega\}$$

Thick enough arithmetic planes are full convex



Arithmetic plane

- irreducible normal vector $N \in \mathbb{Z}^d$
- intercept $\mu \in \mathbb{Z}$
- positive thickness $\omega \in \mathbb{Z}, \omega > 0$

$$P(\mu, N, \omega) := \{x \in \mathbb{Z}^d, \mu \leqslant x \cdot N < \mu + \omega\}$$

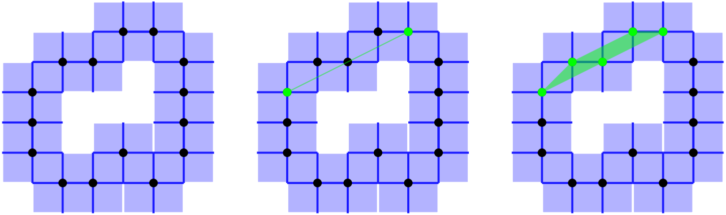
Theorem

Arithmetic planes are fully convex for thickness $\omega \geqslant \|N\|_\infty$.

Tangency

Definition

The digital set $A \subset X \subset \mathbb{Z}^d$ is said to be *k-tangent to X* for $0 \leqslant k \leqslant d$ whenever $\bar{\mathcal{C}}^d_k[\mathrm{cvxh}\,(A)] \subset \bar{\mathcal{C}}^d_k[X]$. It is *tangent to X* if the relation holds for all such k. Elements of A are called *cotangent*.

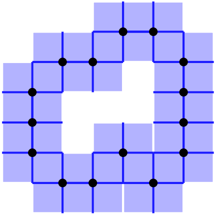


X and $\bar{\mathcal{C}}^d[X]$ tangent tangent

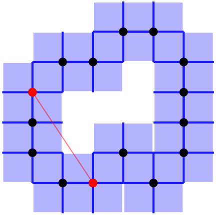
Tangency

Definition

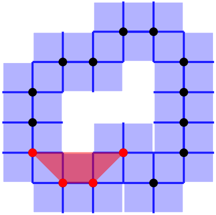
The digital set $A \subset X \subset \mathbb{Z}^d$ is said to be *k-tangent to X* for $0 \leqslant k \leqslant d$ whenever $\bar{\mathcal{C}}^d_k[\mathrm{cvxh}\,(A)] \subset \bar{\mathcal{C}}^d_k[X]$. It is *tangent to X* if the relation holds for all such k. Elements of A are called *cotangent*.



X and $\bar{\mathcal{C}}^d[X]$



not tangent



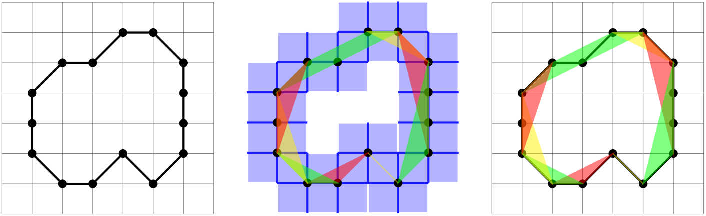
not tangent

Tangential cover

In 2D, maximal fully convex tangent subsets form the classical tangential cover of [Feschet, Tougne 99]

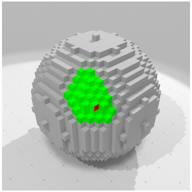
Theorem

When $d = 2$, if C is a simple 2-connected digital contour (i.e. 8-connected in Rosenfeld's terminology), then the fully convex subsets of C that are maximal and tangent are the classical maximal naive digital straight segments.



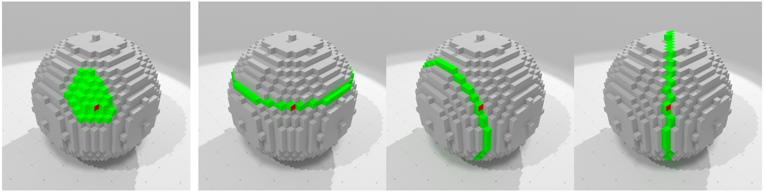
Tangential cover in 3D ? dD ?

- can we define facets of X as inextensible connected pieces of arithmetic planes standard planes along X ?



Tangential cover in 3D ? dD ?

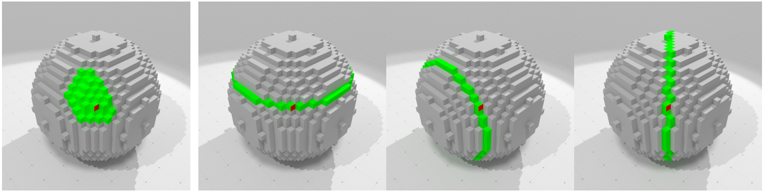
- can we define facets of X as inextensible connected pieces of arithmetic planes standard planes along X ?



- contrarily to 2D, maximal pieces of planes are **not tangent**.
  - there are a lot of inextensible DPS
  - most of them are meaningless

Tangential cover in 3D ? dD ?

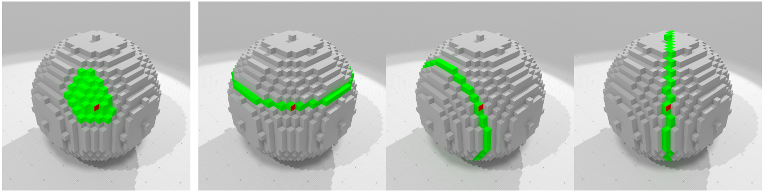
- can we define facets of X as inextensible connected pieces of arithmetic planes standard planes along X ?



- contrarily to 2D, maximal pieces of planes are **not tangent**.
  - there are a lot of inextensible DPS
  - most of them are meaningless
- greedy methods to isolate meaningful ones:
  [Klette, Sun, Coeurjolly, Sivignon, Kenmochi, Provot, Debled-Rennesson, Charrier, L., . . .]

Tangential cover in 3D ? *d*D ?

- can we define facets of X as inextensible connected pieces of arithmetic planes standard planes along X ?



- contrarily to 2D, maximal pieces of planes are **not tangent**.
  - there are a lot of inextensible DPS
  - most of them are meaningless
- greedy methods to isolate meaningful ones:

  [Klette, Sun, Coeurjolly, Sivignon, Kenmochi, Provot, Debled-Rennesson, Charrier, L., . . .]

Tangency extends to *d*D!

Tangent subsets in our sense are indeed tangent to X since their convex hull must lie close to X.

Piecewise linear reversible reconstruction in dD

Let $\mathrm{Del}(X)$ be the Delaunay complex of X.

Definition

The *tangent Delaunay complex* $\mathrm{Del}_{\mathrm{T}}(X)$ to X is the complex made of the cells τ of $\mathrm{Del}(X)$ such that the vertices of τ are tangent to X.

- its boundary is the convex hull when X is fully convex,

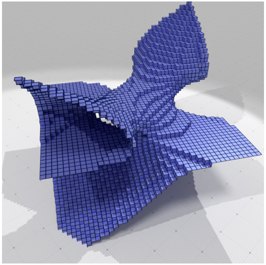
Piecewise linear reversible reconstruction in dD

Let $\mathrm{Del}(X)$ be the Delaunay complex of X.

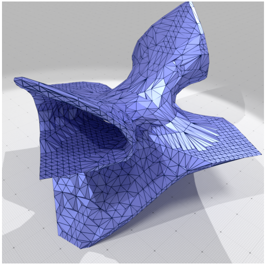
Definition

The *tangent Delaunay complex* $\mathrm{Del}_{\mathrm{T}}(X)$ to X is the complex made of the cells τ of $\mathrm{Del}(X)$ such that the vertices of τ are tangent to X.

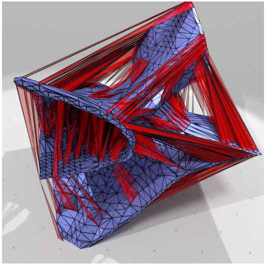
- its boundary is the convex hull when X is fully convex,



Input digital shape X



Reconstruction $\mathrm{Del}_{\mathrm{T}}(X)$



Bad simplices of $\mathrm{Del}(X)$

Theorem

The body of $\mathrm{Del}_{\mathrm{T}}(X)$ is at Hausdorff L_∞-distance 1 to X. $\mathrm{Del}_{\mathrm{T}}(X)$ is a reversible polyhedrization, i.e. $\|\mathrm{Del}_{\mathrm{T}}(X)\| \cap \mathbb{Z}^d = X$.

Conclusion

Pros of full convexity

- natural definition in arbitrary dimension that uses $\mathbb{Z}^d \subset \mathcal{C}^d$
- guarantees connectedness and simple connectedness
- morphological characterization that allows simple convexity check
- thick enough arithmetic planes are fully convex
- entails a consistent definition of tangency
- simple tight and reversible polyhedrization

Cons of full convexity

- $(2^d - 1)$ times slower to check convexity